Moment-based approaches in imaging. Part 1, basic features.

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Moment-based Approaches in Image. Part 1: basic features

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Although the moment theory is well established and widely applied in a number of digital image areas, it remains relatively marginal in medical imaging. Taking a simple example, there is a huge difference between the number of papers published on the use of deformable models and those related to moments. Surprisingly, the formers have a limited spectrum of interest, mainly object segmentation and tracking, two important issues in computer vision of course, when the latters bring major cues in many problems, from reconstruction to detection, from pattern recognition to compression. This does not mean at all that the research conducted on the various moment families and their applications is not active as it will be shown through this series of short papers. Our first objective is to provide to the readers a comprehensive reference source that should be of help for their own research. The readers are also encouraged to look at other surveys, for instance [1-5]. We would like, and it is our second objective, to put some emphasis on some recently studied moments, especially the Tchebichef, Krawtchouk, Racah, dual Hahn, etc. moments, all being orthogonal, an important property in image processing. This first part presents a classification of moments and, rather than entering into theoretical details, it sketches their different expressions. The companion papers will review their properties and the potential contributions they already bring in image.

The pioneering work of Hu in 1962 [6] on moment invariants, moments and moment functions has opened many applications in the image field. Moments can be applied to binary or grey level images, defined in 2D, 3D and higher dimensional space, but also to edges and primitives extracted through a preprocessing stage. A classification is proposed figure 1 going from the Complex Moments (CM), Rotational Moments (RM), Geometric Moments (GM) to Orthogonal Moments (OM). The double arrows mean that each of these moments can be expressed within the other formulation while single arrows denote a sub-class relation. Orthogonal Moments are then decomposed into continuous and discrete families, to which more attention will be paid due to the interesting features they have for image applications. They will be introduced for simplicity using 2D grey level images.

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I. Non-orthogonal Moments

The general two-dimensional (2D) moment definition, using a moment weighting kernel $\varphi_{nm}(x, y)$ (also known as the basis function), and an image intensity function $f(x, y)$, is given by

$$\Psi_{nm} = \iint_{R^2} \varphi_{nm}(x, y) f(x, y) dxdy, \quad n, \; m = 0, \; 1, \; 2, \; \ldots \tag{1}$$

With different basis functions, φ_{nm}, different types of moments can be obtained. In the following, we give a brief description of the above mentioned moments and we point out some of their properties.

A. Geometric moments (GM)

The geometric moments are basically projections of the image function onto the monomials, i.e., $\varphi_{nm}(x, y) = x^n y^m$, the $(n+m)$th order geometric moment, M_{nm}, is defined as

$$M_{nm} = \iint_{R^2} x^n y^m f(x, y) dxdy. \tag{2}$$

The geometric moments are most widely used in image analysis and pattern recognition tasks. This is due essentially to their simplicity, the invariance and geometric meaning of the low order moment values. In fact, the zeroth order moment, M_{00}, represents the total mass of the image. The two first order moments, (M_{10}, M_{01}), provide the position of the center of mass. The second order moments, (M_{20}, M_{11}, M_{02}), can be used to determine several useful image features such as the principal axes, the image ellipse and the radii of gyration [1].

B. Rotational moments (RM)

The rotational moment of order n with repetition m has the kernel $\varphi_{nm}(r, \theta) = r^n e^{jm\theta}$. That is, the 2D rotational moment defined in polar coordinates is given by [7]

$$D_{nm} = \int_0^{2\pi} \int_0^{\infty} r^n e^{jm\theta} f(r, \theta) r dr d\theta, \qquad |m| \leq n, \; n - m = \text{even}. \tag{3}$$

The rotational moments have the nice property to be invariant under image rotation. In fact, if the image is rotated by an angle φ, the relation between the transformed moments and the original moments is

$$D'_{nm} = e^{jm\varphi} D_{nm}. \tag{4}$$

A rotation of φ is thus achieved by a phase change of the rotational moments, so that the magnitude remains invariant. This property makes rotational moments useful descriptors in pattern recognition.

C. Complex moments (CM)

The basis function for the complex moments is $\varphi_{nm}(x, y) = (x + iy)^n(x - iy)^m$, the 2D complex moment of order $(n+m)$ is defined as [8]

$$C_{nm} = \iint_{R^2} (x + iy)^n (x - iy)^m f(x, y) dxdy. \tag{5}$$

The complex moments are related to rotational moments by [1]

$$C_{nm} = D_{n+m,n-m} \tag{6}$$

and thus, the rotation transformation of the image affects only to the phase of the complex moments.

II. Orthogonal Moments (OM)

The geometric moment definition has the form of the projection of $f(x, y)$ onto the monomials $x^n y^m$. Unfortunately, the basis set $\{x^n y^m\}$ is not orthogonal. Consequently, these moments are not optimal with regard to the information redundancy. Moreover, the lack of orthogonality causes the recovery of an image from its geometric moments strongly ill-posed. To overcome the shortcomings associated with geometric moments, Teague [9] suggested the use of the orthogonal moments that are defined in terms of the continuous orthogonal polynomials such as Legendre and Zernike polynomials. Recently, the discrete orthogonal moments (e.g., Tchebichef moments, Krawtchouk moments, Racah moments, and dual Hahn moments) have also been introduced.

A. Continuous orthogonal moments

Legendre moments (LM)

The basis function for Legendre moment is $\varphi_{nm}(x, y) = P_n(x)P_m(y)$ where $P_p(x)$ denotes the pth order of Legendre polynomial. The $(n+m)$th order of Legendre moment, L_{nm}, is defined as [9]

$$L_{nm} = \frac{(2n+1)(2m+1)}{4} \int_{-1}^{1}\int_{-1}^{1} P_n(x)P_m(y) f(x,y)\,dxdy\,. \tag{7}$$

where $P_n(x)$ is the nth order of Legendre polynomial given by

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{n/2} (-1)^k \frac{(2n-2k)!}{k!(n-k)!(n-2k)!} x^{n-2k}\,. \tag{8}$$

Since the Legendre polynomials are orthogonal over the interval [-1, 1], the image $f(x, y)$ can be reconstructed from its moments. Teague derived a simple approximation to the inverse transform for a set of moments through order M given by

$$f(x,y) \approx \sum_{n=0}^{M}\sum_{m=0}^{n} L_{n-m,m} P_{n-m}(x) P_m(y)\,. \tag{9}$$

Zernike moments (ZM)

The Zernike moments use the complex Zernike polynomials as the moment basis set. The 2D Zernike moments, Z_{nm}, of order n with repetition m, are defined in polar coordinates (r, θ) inside the unit circle as [9]

$$Z_{nm} = \frac{n+1}{\pi} \int_0^1\int_0^{2\pi} R_{nm}(r) e^{-jm\theta} f(r,\theta)\, r\,dr\,d\theta, \qquad 0 \le |m| \le n, \quad n-|m| \text{ is even.} \tag{10}$$

where $R_{nm}(r)$ is the nth order of Zernike radial polynomial given by

$$R_{nm}(r)=\sum_{k=0}^{(n-|m|)/2}(-1)^k\frac{(n-k)!}{k!\left[(n-2k+|m|)/2\right]!\left[(n-2k-|m|)/2\right]!}r^{n-2k}\,. \tag{11}$$

Like the rotational moments and the complex moments, the magnitude of the Zernike moments is invariant under image rotation transformation. The image can be reconstructed using a set of moments through order M as

$$f(r,\theta)\approx\sum_{n=0}^{M}\sum_{m}Z_{nm}R_{nm}(r)e^{jm\theta}\,. \tag{12}$$

Pseudo-Zernike moments (PZM)

The 2D pseudo-Zernike moments are based on a set of pseudo-Zernike polynomials that have properties analogue to Zernike polynomials. The pseudo-Zernike polynomials are defined by [10]

$$S_{nm}(r)=\sum_{k=0}^{n-|m|}(-1)^k\frac{(2n+1-k)!}{k!(n-|m|-k)!(n+|m|+1-k)!}r^{n-k}\,,\quad 0\le|m|\le n. \tag{13}$$

Like the Zernike moments, the pseudo-Zernike moments possess the good properties of orthogonality and rotation invariance. The set of pseudo-Zernike moments contains $(M+1)^2$ linearly independent polynomials of order up to M, while Zernike moments have $(M+1)(M+2)/2$ linearly independent polynomials due to the additional constraints of $n-|m|$ being even, therefore the pseudo-Zernike moments have a better feature representation capability. It was also proven that the pseudo-Zernike moments are more robust to image noise than the conventional Zernike moments.

Other kinds of orthogonal moments including Fourier-Mellin moments [11], Chebyshev-Fourier moments (CFM) [12], radial harmonic Fourier [13] and generalized pseudo-Zernike moments (GPZM) [14] have been recently reported in the literature. For more details, we refer the readers to the corresponding references.

B. Discrete orthogonal moments

As stated by Yap et al [15], one common problem with the continuous moments is the discretization error, which accumulates as the order of the moments increases, thus affects the accuracy of the computed moments. To face this problem, a set of discrete orthogonal moments has been recently introduced.

Tchebichef moments (TM)

For a digital image $f(x, y)$ with size $N\times N$, the $(n+m)$th order Tchebichef moments are defined as [4]

$$T_{nm}=\frac{1}{\tilde{\rho}(n,N)\tilde{\rho}(m,N)}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\tilde{t}_n(x)\tilde{t}_m(y)f(x,y)\,, \tag{14}$$

where the scaled Tchebichef polynomials $\{\tilde{t}_n(x)\}$ are given by

$$\tilde{t}_n(x) = \frac{t_n(x)}{\beta(n,N)}, \tag{15}$$

$$t_n(x) = (1-N)_n\, {}_3F_2(-n,-x,1+n;1,1-N;1), \tag{16}$$

$$\tilde{\rho}(n,N) = \frac{\rho(n,N)}{\beta(n,N)^2}, \tag{17}$$

$$\rho(n,N) = (2n)!\binom{N+n}{2n+1}, \qquad p = 0, 1, \ldots, N-1. \tag{18}$$

Here, $\beta(n, N)$ is a suitable constant which is independent of x and ${}_3F_2(\cdot)$ is the hypergeometric function defined as

$${}_3F_2(a_1,a_2,a_3;b_1,b_2;z) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k(a_3)_k}{(b_1)_k(b_2)_k} \frac{z^k}{k!}, \tag{19}$$

with $(a)_k$ the Pochhammer symbol given by

$$(a)_k = a(a+1)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}. \tag{20}$$

The scaled Tchebichef polynomials satisfy the orthogonality property

$$\sum_{x=0}^{N-1} \tilde{t}_n(x)\tilde{t}_m(y) = \delta_{nm}, \qquad 0 \le n, m \le N-1. \tag{21}$$

The orthogonality property leads to the following inverse moment transform

$$f(x,y) = \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} T_{nm}\tilde{t}_n(x)\tilde{t}_m(y). \tag{22}$$

Krawtchouk moments (KM)

The Krawtchouk moment of order $(n+m)$ of an image $f(x, y)$ with size $N\times N$ is defined as [15]

$$Q_{nm} = \sum_{x=0}^{N-1}\sum_{y=0}^{N-1} \tilde{K}_n(x;p_1,N-1)\tilde{K}_m(y;p_2,N-1)f(x,y), \tag{23}$$

where the set of weighted Krawtchouk polynomials $\{\tilde{K}_n(x;p,N)\}$ is defined by

$$\tilde{K}_n(x;p,N) = K_n(x;p,N)\sqrt{\frac{w(x;p,N)}{\rho(n;p,N)}}, \tag{24}$$

$$K_n(x;p,N) = {}_2F_1(-n,-x;-N;1/p), \quad 0 \le x, n \le N-1. \tag{25}$$

$$w(x; p, N) = \binom{N}{x} p^x (1-p)^{N-x}, \qquad 0 < p < 1, \tag{26}$$

$$\rho(n; p, N) = (-1)^n \left(\frac{1-p}{p}\right)^n \frac{n!}{(-N)_n}, \tag{27}$$

Here, ${}_2F_1(\cdot)$ is the hypergeometric function defined as

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \tag{28}$$

The original image $f(x, y)$ can be completely reconstructed using the Krawtchouk moments as

$$f(x,y) = \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} Q_{nm} \widetilde{K}_n(x; p_1, N-1)\widetilde{K}_m(y; p_2, N-1). \tag{29}$$

Racah moments (RAM)

Both Tchebichef and Krawtchouk polynomials are orthogonal on uniform lattice. Recently, Zhu et al [5] introduced a kind of orthogonal polynomials defined on non-uniform lattice, known as Racah polynomials, to form a new set of orthogonal moments. The $(n+m)$th order Racah moment of an image $f(s, t)$ with size $N\times N$ is defined as [5]

$$U_{nm} = \sum_{s=a}^{b-1}\sum_{t=a}^{b-1} \hat{u}_n^{(\alpha,\beta)}(s,a,b)\hat{u}_m^{(\alpha,\beta)}(t,a,b) f(s,t) \qquad n, m = 0, 1, \ldots, L-1, \tag{30}$$

the set of weighted Racah polynomials being defined as

$$\hat{u}_n^{(\alpha,\beta)}(s,a,b) = u_n^{(\alpha,\beta)}(s,a,b)\sqrt{\frac{\rho(s)}{d_n^2}[\Delta x(s-\frac{1}{2})]} \qquad n = 0, 1, \ldots, L-1, \tag{31}$$

$$\rho(s) = \frac{\Gamma(a+s+1)\Gamma(s-a+\beta+1)\Gamma(b+\alpha-s)\Gamma(b+\alpha+s+1)}{\Gamma(a-\beta+s+1)\Gamma(s-a+1)\Gamma(b-s)\Gamma(b+s+1)}, \tag{32}$$

$$d_n^2 = \frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)\Gamma(b-a+\alpha+\beta+n+1)\Gamma(a+b+\alpha+n+1)}{(\alpha+\beta+2n+1)n!(b-a-n-1)!\Gamma(\alpha+\beta+n+1)\Gamma(a+b-\beta-n)}, \tag{33}$$

$$u_n^{(\alpha,\beta)}(s,a,b) = \frac{1}{n!}(a-b+1)_n(\beta+1)_n(a+b+\alpha+1)_n \, {}_4F_3\left(\begin{matrix}-n, \alpha+\beta+n+1, a-s, a+s+1\\ \beta+1, a+1-b, a+b+\alpha+1\end{matrix};1\right) \tag{34}$$

The generalized hypergeometric function ${}_4F_3(\cdot)$ is given by

$${}_4F_3(a_1,a_2,a_3,a_4;b_1,b_2,b_3;z) = \sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k(a_3)_k(a_4)_k}{(b_1)_k(b_2)_k(b_3)_k}\frac{z^k}{k!}, \tag{35}$$

and the parameters a, b, α and β are restricted to

$$-1/2 < a < b, \quad \alpha > -1, \quad -1 < \beta < 2a+1, \quad b = a+N. \tag{36}$$

The orthogonality property of Racah polynomials helps in expressing the image intensity function $f(s, t)$ in terms of its Racah moments. The image reconstruction can be obtained by using the following inverse Racah moment transform

$$f(s,t) = \sum_{n=0}^{L-1}\sum_{m=0}^{L-1} U_{nm}\hat{u}_n^{(\alpha,\beta)}(s,a,b)\hat{u}_m^{(\alpha,\beta)}(t,a,b), \quad s, t = a, a+1, \ldots, b-1, \tag{37}$$

where (s, t) represents the uniform pixel grid of image.

Dual Hahn moments (DHM)

Another set of discrete orthogonal polynomials, defined on non-uniform lattice, known as dual Hahn polynomials, has also been introduced. The $(n+m)$th order dual Hahn moment of an image $f(s, t)$ with size $N\times N$ is defined as [16]

$$W_{nm} = \sum_{s=a}^{b-1}\sum_{t=a}^{b-1} \hat{w}_n^{(c)}(s,a,b)\hat{w}_m^{(c)}(t,a,b)f(s,t), \qquad n, m = 0, 1, \ldots, N-1, \tag{38}$$

where the weighted Racah polynomials are

$$\hat{w}_n^{(c)}(s,a,b) = w_n^{(c)}(s,a,b)\sqrt{\frac{\rho(s)}{d_n^2}[\Delta x(s-\frac{1}{2})]}, \qquad n = 0, 1, \ldots, N-1, \tag{39}$$

$$w_n^{(c)}(s,a,b) = \frac{(a-b+1)_n(a+c+1)_n}{n!}\,{}_3F_2(-n, a-s, a+s+1; a-b+1, a+c+1; 1), \tag{40}$$

$$\rho(s) = \frac{\Gamma(a+s+1)\Gamma(c+s+1)}{\Gamma(s-a+1)\Gamma(b-s)\Gamma(b+s+1)\Gamma(s-c+1)}, \tag{41}$$

$$d_n^2 = \frac{\Gamma(a+c+n+1)}{n!(b-a-n-1)!\Gamma(b-c-n)}, \quad n = 0, 1, \ldots, N-1. \tag{42}$$

with the hypergeometric function ${}_3F_2(\cdot)$ given by equation (19) and the parameters a, b and c restricted to

$$-1/2 < a < b, \quad |c| < 1+a, \quad b = a+N. \tag{43}$$

III. Conclusion

This first paper was aimed at providing the basic formulations of moments, a classification and an introductory bibliography. The moment-based approaches have a number of interesting features. They have a wide range of orthogonal and non-orthogonal basis functions and are simple to compute whatever the order required. The image sampling can be either rectangular or polar, based on uniform or non-uniform lattices. Their optimal choice to deal with a given problem is however not obvious according to the requirements to face. There are of course many other issues to address. Among the important properties to consider there are (i) the invariance to scale, translation, and orientation, etc. (ii) the robustness to degradations, noise, to changing conditions (illumination) or blurring and (iii) to object variations (multiple appearances, occlusions and deformations). Moment computations have also a cost that is sometimes considered too high for certain applications: this explains that a special attention has been devoted to acceleration techniques and VLSI implementations. These important aspects for computer vision at large and the most meaningful works in medical imaging will be examined in the next papers.

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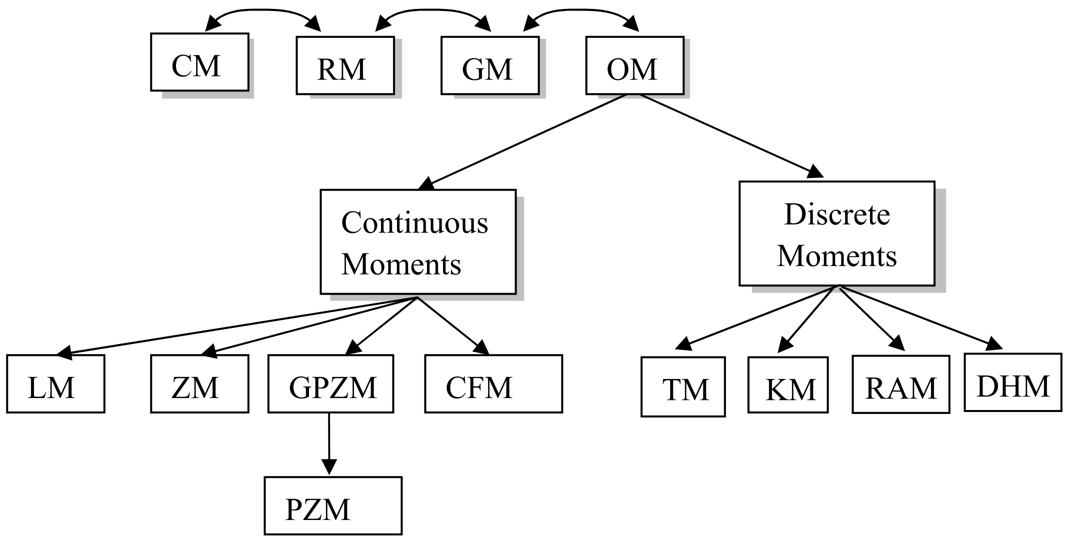

Fig. 1 A classification of family of moments with:
CM: Complex moments; RM: Rotational Moments; GM: Geometric Moments;
OM: Orthogonal Moments; LM: Legendre Moments; ZM: Zernike Moments;
PZM: Pseudo-Zernike Moments; GPZM: Generalized pseudo-Zernike Moments;
CFM: Chebychev-Fourier Moments; TM: Tchebichef Moments;
KM: Krawtchouk Moments; RAM: Racah Moments; DHM: Dual Hahn Moments